Direct and Inverse Lambert Transformations
Based upon Bomford (1980)

Alan P. Vonderohe

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Introduction

This document describes mathematics for direct and inverse conformal Lambert map projection transformations, based upon the functions presented in

Bomford, G., (1980), Geodesy, 4th Edition, Oxford University Press, NY.

Indicated page numbers refer to the Bomford text.

Bomford does not explicitly provide all the mathematics presented here. For example, in one case, Bomford notes that a look-up table is often used, but the text does not present the look-up table. The author (Vonderohe) has developed the mathematics necessary for generating the look-up table and has imbedded them in this document. It is not necessary to generate the look-up table. Rather the functions can merely be used as components of the transformation algorithms.

We will first present ellipsoidal parameters and functions that are common to both the direct and inverse transformations. We will then present the Lambert parameters, functions used in both the direct and inverse transformations, the direct transformation, and, finally, the inverse transformation. The Lambert parameters do not include two standard parallels along which the scale factor is equal to one. Rather, they include one central parallel along which the scale factor is arbitrary. In that way, Lambert projections that do not intersect the ellipsoid can be developed.

NOTE: All latitudes (ϕ) are positive north of the equator. All longitudes (λ) are negative west of Greenwich.

Ellipsoid Parameters

a = semi-major axis.

b = semi-minor axis.

Ellipsoid Functions

$$e^2 = \frac{a^2 - b^2}{a^2}$$ e is the eccentricity (page 646).

$$\varepsilon = e'^2 = \frac{a^2 - b^2}{b^2}$$ e' is the second eccentricity (page 646).

Lambert Transformations

Parameters

ϕ_o = Latitude of central parallel and coordinate origin.
λ_o = Longitude of central meridian and coordinate origin.
m_o = Scale along central parallel.
N_o = False northing of coordinate origin.
E_o = False easting of coordinate origin.

Lambert Functions Used in Both Direct and Inverse Transformations

$$\nu_o = \frac{a}{\sqrt{1-e^2\sin^2\phi_o}}$$ = Ellipsoidal radius of curvature in the meridian at ϕ_o. (page 647)

$$\rho_o = \frac{a(1-e^2)}{(1-e^2\sin^2\phi_o)^{\frac{3}{2}}}$$ = Ellipsoidal radius of curvature in the prime vertical at ϕ_o. (page 647)

$$r_o = \frac{m_o \nu_o}{\tan\phi_o}$$ = Mapping radius at coordinate origin. (page 187)

$$A_o = 1-\frac{e^2}{4}-\frac{3e^4}{64}-\frac{5e^6}{256}$$ (page 648)

$$A_2 = \frac{3}{8}\left(e^2+\frac{e^4}{4}+\frac{15e^6}{128}\right)$$ (page 648)

$$A_4 = \frac{15}{256}\left(e^4+\frac{3e^6}{4}\right)$$ (page 648)

$$A_6 = \left(\frac{35}{3072}\right)e^6$$ (page 648)

$$M_o = a(A_0\phi_o - A_2\sin 2\phi_o + A_4\sin 4\phi_o - A_6\sin 6\phi_o)$$ = Meridional arc from ϕ_o to equator (ϕ_o in radians). (page 647)

Direct Transformation ($\phi,\lambda \rightarrow N,E$)

$$M_\phi = a(A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi)$$ = Meridional arc from ϕ to equator (ϕ in radians). (page 647)

$$M = M_\phi - M_o$$ = Meridional arc from ϕ to ϕ_o. M is negative if $M_\phi < M_o$.

$$M_3 = \frac{M^3}{6\rho_o \nu_o}$$

$$M_4 = \frac{M^4 \tan\phi_o (1 - 4\varepsilon \cos^2 \phi_o)}{24\rho_o \nu_o^2}$$

$$M_5 = \frac{M^5 (5 + 3\tan^2 \phi_o - 3\varepsilon - \varepsilon \cos^2 \phi_o)}{120\rho_o^2 \nu_o^2}$$

$$M_6 = \frac{M^6 \tan\phi_o (7 + 4\tan^2 \phi_o)}{240\rho_o^2 \nu_o^3}$$

$$M_7 = \frac{M^7 (60\tan^4 \phi_o + 180\tan^2 \phi_o + 61)}{5040\rho_o^3 \nu_o^3}$$

$s = m_o (M + M_3 + M_4 + M_5 + M_6 + M_7)$ = Meridional arc from ϕ to ϕ_o scaled to the projection. s is negative if $M_\phi < M_o$. (page 188)

$\gamma = (\lambda - \lambda_o)\sin\phi_o$ = Convergence at λ. (page 189)

$\Delta E = (r_o - s)\sin\gamma$ (page 188)

$E = E_o + \Delta E$

$N = N_o + s + \Delta E \tan\left(\frac{\gamma}{2}\right)$ (page 188)

Inverse Transformation ($N, E \rightarrow \phi, \lambda$)

$\gamma = \tan^{-1}\left(\frac{E - E_o}{r_o - N + N_o}\right)$ (page 188)

$\lambda = \left(\frac{\gamma}{\sin\phi_o}\right) + \lambda_o$ (page 188)

$s = N - N_o - (E - E_o)\tan\left(\frac{\gamma}{2}\right)$ (page 188)

NOTE: Here Bomford indicates that ϕ is looked up in a table. We have no table, but we have the functions that would generate such a table. We will not create the table. Instead, we will solve the functions backwards for ϕ. First we will solve equation 2.99 on page 188 backwards for M. We will linearize the equation using Taylor's series and then drop all the higher-order terms. That will give us

$$\delta M = \frac{-F_a}{\left(\frac{\partial F}{\partial M}\right)_a}$$

where δM is a correction to the latest approximation for M, F_a is the function in equation 2.99 evaluated at the latest approximation for M, and $\left(\frac{\partial F}{\partial M}\right)_a$ is the partial derivative of F with respect to M evaluated at the latest approximation for M. An initial approximation for M can be found by

$$M_a = \frac{s}{m_o}$$

M_a should be updated as $M_a = M_a + \delta M$ and iterations should continue until $\delta M <$ 0.00005m. During each iteration,

$$M_{3_a} = \frac{M_a^3}{6\rho_o \nu_o}$$

$$M_{4_a} = \frac{M_a^4 \tan\phi_o (1 - 4\varepsilon \cos^2 \phi_o)}{24\rho_o \nu_o^2}$$

$$M_{5_a} = \frac{M_a^5 (5 + 3\tan^2 \phi_o - 3\varepsilon - \varepsilon \cos^2 \phi_o)}{120\rho_o^2 \nu_o^2}$$

$$M_{6_a} = \frac{M_a^6 \tan\phi_o (7 + 4\tan^2 \phi_o)}{240\rho_o^2 \nu_o^3}$$

$$M_{7_a} = \frac{M_a^7 (60\tan^4 \phi_o + 180\tan^2 \phi_o + 61)}{5040\rho_o^3 \nu_o^3}$$

$$-F_a = \left(\frac{s}{m_o}\right) - M_a - M_{3_a} - M_{4_a} - M_{5_a} - M_{6_a} - M_{t_a}$$

$$\left(\frac{\partial F}{\partial M}\right)_a = 1 + \frac{3M_{3_a}}{M_a} + \frac{4M_{4_a}}{M_a} + \frac{5M_{5_a}}{M_a} + \frac{6M_{6_a}}{M_a} + \frac{7M_{7_a}}{M_a}$$

After convergence on M:

$$M_\phi = M_o + M$$

Now, we solve equation A.68 on page 647 backwards for ϕ in the same way that we solved equation 2.99 backwards for M:

$$\delta\phi = \frac{-G_a}{\left(\frac{\partial G}{\partial \phi}\right)_a}$$

where $\delta\phi$ is a correction to the latest approximation for ϕ, G_a is the function in equation A.68 evaluated at the latest approximation for ϕ, and $\left(\frac{\partial G}{\partial \phi}\right)_a$ is the partial derivative of G with respect to ϕ evaluated at the latest approximation for ϕ. An initial approximation for ϕ can be found by

$$\phi_a = \frac{M_\phi}{aA_o} \text{ (radians)}$$

ϕ_a should be updated as $\phi_a = \phi_a + \delta\phi$ and iterations should continue until $\delta\phi < 0.0000005$ seconds of arc. During each iteration,

$$-G_a = M_\phi - a(A_0\phi_a - A_2 \sin 2\phi_a + A_4 \sin 4\phi_a - A_6 \sin 6\phi_a)$$

$$\left(\frac{\partial G}{\partial \phi}\right)_a = a(A_0 - 2A_2 \cos 2\phi_a + 4A_4 \cos 4\phi_a - 6A_6 \cos 6\phi_a)$$